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Supersymmetric color superconductivity*

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Abstract: Recent interest in novel phases in high density QCD motivates the study of high density supersymmetric QCD (SQCD), where powerful exact results for supersymmetric gauge theories can be brought to bear in the strongly coupled regime. We begin by describing how a chemical potential can be incorporated into a supersymmetric theory as a spurion vector superfield. We then study supersymmetric SU(N_c) gauge theories with N_f flavors of quarks in the presence of a baryon chemical potential μ, and describe the global symmetry breaking patterns at low energy. Our analysis requires $\mu < \Lambda$ and is thus complementary to the variational approach that has been successful for $\mu \gg \Lambda$. We find that for $N_f < N_c$ a modified U$(1)_B$ symmetry is preserved, analogous to the non-supersymmetric 2SC phase, whereas for $N_f = N_c$ there is a critical chemical potential above which the U$(1)_B$ is broken, as it is in the non-supersymmetric CFL phase. We further analyze the cases with $N_c + 1 \le N_f < \frac{3}{2}N_c$ and find that baryon number is broken dynamically for $\mu > \mu_c$. We also give a qualitative description of the phases in the 'conformal window', $\frac{3}{2}N_c < N_f < 3N_c$, at finite density.

Keywords: Supersymmetry and Duality, -REM-Related Astrophysics (Neutron Stars, Supernovae etc.), QCD.

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JHEP03(2004)049

Contents

1. Introduction

In the past several years there has been a lot of interest in QCD at high density where novel phases of matter are predicted to appear (see the reviews [1, 2] and references therein). High densities imply that the Fermi surface lies at a high energy scale for QCD, which means it is weakly coupled. The tools of BCS theory can then be used to demonstrate the instability of the Fermi surface and the existence of lower energy condensates which exhibit interesting symmetry breaking patterns such as color superconductivity and color-flavor locking. These new phases might eventually be observable in compact star dynamics. One important consequence of high density QCD is the dynamical breakdown of baryon number for three or more quark flavors.

Because of asymptotic freedom previous treatments of finite density QCD were limited to very high density where the physics is perturbative. It would be interesting to find QCD-like theories that are calculable even at low densities where the Fermi surface lies within the strongly coupled regime. During the last decade, there has been significant progress uncovering exact results in supersymmetric gauge theories. In particular, the low energy description of supersymmetric-QCD (SQCD) has been described for all numbers of colors N_c and fundamental matter flavors, N_f. (For reviews, see [3, 4].)

In this paper we use the exact results provided by supersymmetry to study the effects of a baryon chemical potential on the vacuum of supersymmetric QCD. The tight restrictions imposed by supersymmetry allow us to reduce the set of possible vacua and symmetry breaking patterns. A clear limitation of this approach is that the symmetry breaking patterns of supersymmetric and non-supersymmetric QCD are different even in the absence of a chemical potential, so we cannot expect them to be the same in the presence of a chemical potential. On the other hand, one can ask a less specific question: is baryon number broken dynamically?

The main result of our study is that indeed this question has the same answer in both non-supersymmetric and supersymmetric theories for $N_f \leq \frac{3}{2}N_c$ as described below, giving additional support to the variational results using the BCS states. We emphasize that the dynamical breakdown of $\mathrm{U}(1)_B$ we have studied differs from the simple Bose-Einstein condensation which necessarily occurs in a theory containing scalars when the chemical potential exceeds the mass of the scalar. We show explicitly that baryon number is broken as a consequence of the strong gauge dynamics at a lower chemical potential. It is also intriguing that baryon number can be dynamically broken even if the chemical potential is much smaller than the dynamical scale.

The results for the symmetry breaking patterns for non-supersymmetric QCD in the presence of a large chemical potential have been studied in [5, 6] for $N_c = 3$ and are briefly summarized in table 1. Note that for $N_f < N_c$ a global U(1) remains unbroken. For $N_f = 2$ this U(1) is a combination of the original baryon number and a diagonal color generator. However, for $N_f \geq N_c$ baryon number is spontaneously broken. Below we will compare these results with the possible symmetry breaking patterns in SQCD for $N_c \geq 3$ and various numbers of flavors.

We want to study the dynamics of supersymmetric QCD in the presence of a chemical potential μ which explicitly breaks supersymmetry. One main difference between the supersymmetric and non-supersymmetric gauge theories is of course the presence of scalar quarks. If the chemical potential is larger than the scalar mass the squarks immediately undergo the standard Bose-Einstein condensation. Since we are interested in the effect of strong gauge dynamics, we will add a stabilizing mass to prevent such condensation. Such a mass can be either supersymmetric or SUSY-breaking. We will study both cases in turn.

N_f	Unbroken Global Symmetry
1	$U(1)_B$
2	$SU(2)_L \times SU(2)_R \times U(1)_B$ (2SC)
3	$SU(3)_{L+R+c}$ (CFL)
4	$SU(2)_V \times SU(2)_V \times SU(2)_A$
5	$SU(2)_{L+R}$
6	$SU(3)_{L+R+c} \times SU(2)_{L+R+c} \times U(1)_V \times U(1)_A$

Table 1: Unbroken subgroup of the original $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ global symmetry for QCD with $N_c = 3$ according to [5, 6].[1]

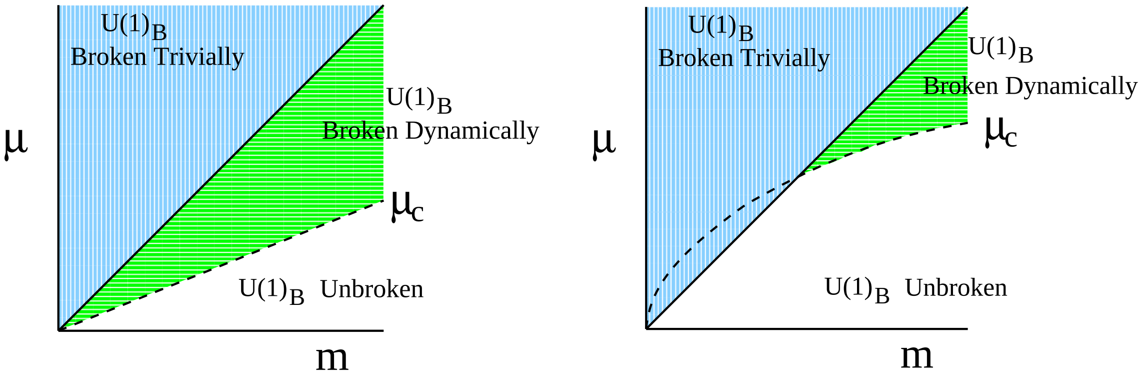

Figure 1: Schematic phase diagrams in the μ-m plane where m is the supersymmetric mass. The diagram on the left shows the situation for $N_f = N_c$ and the diagram on the right shows the situation for $N_c + 1 \leq N_f < 2N_c$. The dashed line is the critical chemical potential. These diagrams are only valid for $\mu < \Lambda$.

Throughout we will assume that $\mu < \Lambda$ so that the chemical potential can be treated as a small perturbation compared to the strong supersymmetric gauge dynamics that have been well studied. This is a very different regime from the range of validity of the BCS variational method in color superconductivity, namely $\mu \gg \Lambda$. In the latter case, the dynamics is weakly coupled and the analysis is under control. On the other hand, in our supersymmetric analysis the dynamics is strongly coupled. The added constraints from supersymmetry allow us to draw interesting conclusions about symmetry breaking patterns. Our analysis is therefore complementary to the variational method of non-supersymmetric color superconductivity. However, we cannot say anything about gauge non-invariant quantities such as $\langle qq \rangle$. Furthermore, we have assumed that the vacuum is translationally invariant, which means we are ignoring the possibility of a crytalline phase [7].

Here we briefly summarize our results. We find that baryon number is unbroken when $N_f < N_c$, but that there is a critical chemical potential above which baryon number is broken when $N_c \leq N_f < \frac{3}{2}N_c$. This pattern matches the non-supersymmetric results. The dependence of the critical chemical potential on the stabilizing mass can be of two qualitatively different types, as illustrated in figure 1. The situation for $N_f = N_c$ is shown

[1] The result for $N_f = 4$ in the published version of [5] is incorrect, but has been corrected in the e-print version. For $N_f = 6$ the remaining $U(1)_V \times U(1)_A$ symmetry is a subgroup of $SU(6)_L \times SU(6)_R$ and is thus physically distinct from $U(1)_B$.

JHEP03(2004)049

in the left diagram, where μ_c is always less than the stabilizing mass m. The situation for more flavors is illustrated in the right diagram, where we see that there is a minimum value of the stabilizing mass that allows for a nontrivial critical chemical potential. In all cases we require $\mu < \Lambda$ so that our supersymmetric analysis is valid.

The outline of the paper is as follows. In the next section we review the formalism for constructing an effective lagrangian in the presence of a chemical potential. In section 3 we adapt this formalism to SQCD. In sections 4–7 we determine the global symmetries of SQCD with various numbers of flavors, and in section 8 we conclude. Appendix A contains a simple and explicit example in quantum mechanics, as a check on the method of including a chemical potential, while appendix B reviews the exact results for soft masses in SUSY theories.

2. Relativistic Bose-Einstein condensation

Let us begin by reviewing the relativistic formulation of Bose-Einstein condensation for a non-supersymmetric scalar field theory. A nice description is given in [8, 9]. There are two purposes to this discussion. One is to show that we can regard the chemical potential as the time component of a fictitious gauge field of the $\mathrm{U}(1)_B$ symmetry at zero temperature. The other is to find the criterion for the $\mathrm{U}(1)_B$ *not* to be immediately broken in the presence of a chemical potential so we can study the *dynamical* breakdown.

The partition function in the grand canonical ensemble with nonzero chemical potential can be calculated using the following path integral:

$$\begin{aligned} Z &= \mathrm{Tr}\ e^{-\beta(H-\mu N)} \\ &= C\int \mathcal{D}\pi^\dagger\mathcal{D}\pi \int \mathcal{D}\phi^\dagger\mathcal{D}\phi e^{\int_0^\beta d\tau \int d^3x \left[i\pi\frac{\partial\phi}{\partial\tau}+i\pi^\dagger\frac{\partial\phi^\dagger}{\partial\tau}-\mathcal{H}(\pi,\phi)+\mu\mathcal{N}(\pi,\phi)\right]} \end{aligned} \tag{2.1}$$

where $\mathcal{N}$ is the time-component of some conserved current. We consider the case of a complex scalar field, where the hamiltonian is

$$\mathcal{H} = \pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi \tag{2.2}$$

and the conserved current is $J_\mu = i(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger)$. Thus $\mathcal{N} = i(\phi^\dagger\pi^\dagger - \phi\pi)$. The integrand of the exponent in the path integral can be rewritten

$$\begin{aligned} i\left(\pi^\dagger\partial_\tau\phi^\dagger + \pi\partial_\tau\phi\right) - \left(\pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi\right) + i\mu\left(\pi^\dagger\phi^\dagger - \pi\phi\right) = \\ = -\left(\pi^\dagger - i(\partial_\tau-\mu)\phi\right)\left(\pi - i(\partial_\tau+\mu)\phi^\dagger\right) - \\ -(\partial_\tau+\mu)\phi^\dagger(\partial_\tau-\mu)\phi - \nabla\phi^\dagger\cdot\nabla\phi - m^2\phi^\dagger\phi\,. \end{aligned} \tag{2.3}$$

Performing the functional integration over π and $\pi^\dagger$ leads to the following expression for the partition function:

$$Z = C'\int \mathcal{D}\phi^\dagger\mathcal{D}\phi\ e^{-\int_0^\beta d\tau\int d^3x\left[(\partial_\tau+\mu)\phi^\dagger(\partial_\tau-\mu)\phi+\nabla\phi^\dagger\cdot\nabla\phi+m^2\phi^\dagger\phi\right]} \tag{2.4}$$

which can be analytically continued to Minkowski space to yield an effective lagrangian:

$$C' \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \; e^{i\int dt \int d^3x \left[(\partial_t + i\mu)\phi^\dagger(\partial_t - i\mu)\phi - \nabla\phi^\dagger \cdot \nabla\phi - m^2\phi^\dagger\phi\right]} \equiv C' \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \; e^{i\int dt \int d^3x \, \mathcal{L}_{\rm eff}} . \tag{2.5}$$

It is important to note that $\mathcal{L}_{\rm eff}$ is not simply $\mathcal{L} + \mu\mathcal{N}$, since $\mathcal{N}$ is a function of π in addition to ϕ. Instead,

$$\mathcal{L}_{\rm eff} = \partial^\nu\phi^\dagger\partial_\nu\phi + i\mu\left(\phi^\dagger\partial_t\phi - \phi\partial_t\phi^\dagger\right) - (m^2 - \mu^2)\phi^\dagger\phi . \tag{2.6}$$

The term linear in μ is the expected $\mu\mathcal{N}$ contribution. The term quadratic in μ arises from the modification of the conjugate momenta $\pi = \dot{\phi}^\dagger + i\mu\phi$. Here we see immediately that for $m^2 > 0$ there is a critical chemical potential, $|\mu_c| = m$ such that for $|\mu| > |\mu_c|$ the scalar potential is unstable and Bose-Einstein condensation takes place. This can also be seen explicitly by using the result for the transition temperature for ϕ^4 theory as in [8]:

$$T_c^2 = \frac{3}{\lambda}(\mu^2 - m^2) . \tag{2.7}$$

This demonstrates again that condensation occurs for $\mu^2 > m^2$.

In appendix A we test this method of adding a chemical potential in quantum mechanics. We do this by comparing the partition function computed directly with the one obtained using the effective lagrangian derived above. The system we consider is a two-dimensional harmonic oscillator with an added source term for z-component of angular momentum which plays the role of the conserved current.

The purpose of the preceding exercise was to demonstrate that the inclusion of a chemical potential in the lagrangian amounts to modifying the time derivative $\partial_t \to \partial_t - i\mu$, which is equivalent to the addition of a non-dynamical gauge field that acquires a nonzero vacuum expectation value (VEV) for its time component. Explicitly, the effective lagrangian of eq. (2.6) can be rewritten as

$$\mathcal{L}_{\rm eff} = (D^\mu\phi)^\dagger D_\mu\phi - m^2\phi^\dagger\phi \tag{2.8}$$

where

$$D_\mu\phi = (\partial_\mu - igA_\mu)\phi \qquad \text{and} \qquad \langle A_\mu\rangle = \left(\frac{\mu}{g}, 0, 0, 0\right) . \tag{2.9}$$

It can be shown that the same "covariant derivative" gives the correct effective lagrangian for a theory including fermions and is therefore suitable for use in supersymmetric theories. The advantage of this formalism is that the coupling of the chemical potential to the low energy degrees of freedom is determined by their $\mathrm{U}(1)_B$ quantum numbers.

3. SQCD with a baryon chemical potential

We will study supersymmetric QCD which is defined to be an $\mathrm{SU}(N_c)$ super Yang-Mills theory with N_f flavors of chiral superfield quarks Q_i in the fundamental N_c representation of the gauge group, along with N_f flavors of chiral "anti-quarks" $\overline{Q}_i$ in the conjugate representation. This theory has the anomaly free global symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times$

JHEP03(2004)049

$\mathrm{U}(1)_B \times \mathrm{U}(1)_R$. A frequently appearing quantity is the one-loop beta function coefficient $b_0 = 3N_c - N_f$. Much is already known about the infrared limit of SQCD, which depends on the relative numbers of flavors and colors. For $N_f \leq N_c + 1$ the low energy description is in terms of composite mesons and baryons, whereas for higher N_f it is in terms of mesons and dual quarks.

In this paper we study SQCD with a baryon chemical potential. The regular SQCD lagrangian is defined at a high UV scale, where we also add chemical potential terms associated with baryon number for both fermions and bosons. In order to most easily apply the known exact results for supersymmetric gauge theories mentioned above [3], we will follow the discussion in the previous section and incorporate the added chemical potential as a background (fictitious) $\mathrm{U}(1)_B$ super-gauge field that has received the appropriate vacuum expectation value. The resulting UV lagrangian is

$$\mathcal{L}_{UV} = \int d^4\theta \left(Q_i^\dagger e^{g_B V_B} Q_i + \overline{Q}_i^\dagger e^{-g_B V_B} \overline{Q}_i \right) + \frac{1}{g^2} \int d^2\theta \left(\mathcal{W}_\alpha \mathcal{W}^\alpha + \text{h.c.} \right) \tag{3.1}$$

where

$$\langle V_B \rangle = \bar{\theta} \sigma^\nu \theta \langle A_\nu \rangle \qquad \text{with} \qquad \langle A_\nu \rangle = \left(\frac{\mu}{g_B}, 0, 0, 0 \right). \tag{3.2}$$

In eq. (3.1) the $\mathrm{SU}(N_c)$ gauge coupling has been suppressed and the second term gives the gauge kinetic terms for the $\mathrm{SU}(N_c)$ gauge bosons but not for the $\mathrm{U}(1)_B$, which remains non-dynamical.

Note that the VEV in eq. (3.2) explicitly breaks supersymmetry, so one may worry that this invalidates the powerful supersymmetric results, namely that color is confined and that the low energy degrees of freedom can be described by mesons, baryons, or dual quark superfields. We will therefore consider a chemical potential that is small compared to the dynamical scale Λ. In such a case we can treat the chemical potential as a perturbation to the supersymmetric dynamics.

As we have seen above, the chemical potential μ contributes a tachyonic mass term to scalar potentials as in eq. (2.6). For SQCD with massless quark superfields, this means that squark condensation is immediately favored. Because we are interested in studying the effect of strong dynamics on condensation and symmetry breaking, we will add a stabilizing mass for the squarks that returns the squark VEV to the origin. From this stable UV theory we can move on to investigate the symmetry breaking patterns that are triggered in the IR description of the theory by strong dynamics.

We will consider both supersymmetric and SUSY-breaking masses to stabilize the theory in the UV. A supersymmetric mass term appears in the superpotential:

$$W_{\text{mass}} = \sum_{ij} m_{ij} Q_i \overline{Q}_j = \mathrm{Tr}(mM) \,. \tag{3.3}$$

In order to preserve the most global symmetry, we take the SUSY mass to have the form $m_{ij} = m\delta_{ij}$, which explicitly breaks the chiral flavor symmetry to the diagonal vector subgroup and leaves $\mathrm{U}(1)_B$ intact. Stability further requires $\mu < m$. Such supersymmetric mass terms are nice because they do not damage the exact results of SQCD.

If one wishes to maintain the full global symmetry, the stabilizing masses must break SUSY. A supersymmetry-breaking soft mass may be added to the lagrangian:

$$\mathcal{L}_{\rm mass} = -\widetilde{m}^2 \left(Q^\dagger Q + \overline{Q}^\dagger \overline{Q} \right), \tag{3.4}$$

where the Q's here represent the scalar components of the respective superfields. Again, stability requires $\mu < \widetilde{m}$. Since we rely on many exact results that are only valid when the theory is nearly supersymmetric, we will also require $\widetilde{m} \ll \Lambda$. However, the presence of soft masses for the squarks in the UV theory may alter the low energy potential for mesons and baryons (or dual quarks). This question was addressed in [10] and [12] using spurion arguments (and from a different approach in [13]). The main observation is that scalar soft-masses can be incorporated in a superfield $\mathcal{Z}$ whose couplings are determined by an anomalous $\mathrm{U}(1)_A$ symmetry and RG-invariance. Results for softly broken SUSY are reviewed in the following sections and in greater detail in appendix B.

We are now ready to investigate the effects of adding stabilizing masses and a chemical potential to a UV SQCD theory. In each of the theories below we will first determine the symmetry pattern in the absence of a chemical potential, and then check whether adding a chemical potential breaks symmetries. We are be particularly interested in comparing $\mathrm{U}(1)_B$ and its breaking pattern to the non-supersymmetric case. Since the IR degrees of freedom depend on the number of flavors, we will consider separate cases in the following order: $N_f = N_c$ (quantum modified moduli space), $N_f < N_c$ (Affleck-Dine-Seiberg superpotential), $N_f = N_c + 1$ (confinement without chiral symmetry breaking), and $N_c + 2 \leq N_f < 3N_c$ (dual picture).

4. $N_f = N_c$

When the number of flavors and colors are equal the infrared description is in terms of gauge invariant chiral superfields called "mesons" $M_{ij} = Q_i^a \overline{Q}_j^a$ and "baryons" $B = \epsilon_{i_1 \cdots i_{N_f}} Q_{i_1} \cdots Q_{i_{N_f}}$ and $\overline{B} = \epsilon_{i_1 \cdots i_{N_f}} \overline{Q}_{i_1} \cdots \overline{Q}_{i_{N_f}}$. Here the suppressed color indices are completely antisymmetrized. The mesons transform as $(N_f, \overline{N}_f)$ under the flavor symmetries, whereas the baryons are flavor singlets. The meson and baryon superfields satisfy a constraint that is modified quantum mechanically and can be implemented in a dynamical superpotential with a Lagrange multiplier superfield X:

$$W = \frac{X}{\Lambda^{2N_c - 2}} \left(\det M - B\overline{B} - \Lambda^{2N_c} \right), \tag{4.1}$$

where Λ is the scale of the $\mathrm{SU}(N_c)$ theory. The equation of motion for X enforces the constraint.

We will now add stabilizing masses. In the following subsection we will add supersymmetric masses. In section 4.2 we will add soft masses and discover that we need to choose a vacuum to expand around. In 4.2.1 and 4.2.2 we will consider two such vacua in turn.

4.1 Supersymmetric mass

As discussed above, the theory with massless squarks in the UV is immediately unstable upon the introduction of a chemical potential. We will first consider supersymmetric mass to stabilize the theory in the UV. When the supersymmetric mass term in eq. (3.3) is added to the superpotential (4.1) the F-term equations force the baryons to vanish, but yield the VEVs

$$\langle M_{ij}\rangle = (m^{-1})_{ij}\left(\Lambda^{b_0}\det m\right)^{1/N_c} \qquad \text{and} \qquad \langle X^{N_c}\rangle = -\frac{\det m}{(\det M)^{N_c-1}}\,. \tag{4.2}$$

The VEV for X then manifests itself as a mass term for the baryons. The low energy lagrangian for the baryons is thus

$$\mathcal{L} = \int d^2\theta d^2\bar\theta \frac{c_0}{\Lambda^{2N_c-2}}(B^\dagger e^{N_c V_B} B + \overline{B}^\dagger e^{-N_c V_B}\overline{B}) + \int d^2\theta \frac{m}{\Lambda^{N_c-1}}\overline{B}B \tag{4.3}$$

where c_0 is an unknown coefficient in the Kähler potential which is estimated in naive dimensional analysis to be $\mathcal{O}(1)$. We've shown that at zero chemical potential $\mathrm{U}(1)_B$ is conserved and the chiral symmetry is broken to its vector subgroup.

We will now show that this changes as a chemical potential is added. After canonically normalizing B and $\overline{B}$ and inserting the VEV of V_B from eq. (3.2) to incorporate the chemical potential, the squared mass of the baryons is

$$m^2_{B,\overline{B}} = \frac{m^2}{c_0} - N_c^2\mu^2\,, \tag{4.4}$$

which means there is a critical chemical potential

$$\mu_c^2 = \frac{m^2}{c_0 N_c^2}\,, \tag{4.5}$$

such that for $\mu > \mu_c$ the baryon mass-squared becomes negative, drawing the baryon VEVs away from the origin. We cannot determine exactly where the baryons stabilize, but due to the presence of the SUSY masses we know that the UV theory is stable, which means the baryon VEVs should be of order Λ. Thus we conclude that as long as $c_0 N_c^2 > 1$ there are values of μ such that $\mu_c < \mu < m$ where our approximations are valid and the $\mathrm{U}(1)_B$ symmetry is spontaneously broken. This is depicted in the left diagram of figure 1.

4.2 SUSY-breaking mass

Instead of adding supersymmetric mass terms, we now consider the addition of soft SUSY-breaking mass terms to stabilize the UV theory. In order to determine the potential for the low energy degrees of freedom due to SUSY breaking, we will adopt a dimensionless convention for the fields [12]

$$\widehat{M} \equiv \frac{M}{\Lambda_h^2} \qquad \widehat{B} \equiv \frac{B}{\Lambda_h^{N_c}} \qquad \widehat{\overline{B}} \equiv \frac{\overline{B}}{\Lambda_h^{N_c}}\,. \tag{4.6}$$

The constraint then takes the convenient form

$$\det\widehat{M} - \widehat{B}\widehat{\overline{B}} = 1\,. \tag{4.7}$$

The fields $\widehat{M}$, $\widehat{B}$ and $\widehat{\overline{B}}$ are dimensionless and are uncharged under both the anomalous $\mathrm{U}(1)_A$ symmetry described in appendix B and the anomaly free $\mathrm{U}(1)_R$. This means we cannot use these symmetries to constrain the theory and cannot determine the IR soft masses exactly. However, we will still be able to maintain some control over the sign of the baryon masses and thus rule out some symmetry preserving vacua due to their instability.

The flavor and $\mathrm{U}(1)_B$ symmetries constrain the Kähler potential to be a some function K of real and invariant combinations of the fields

$$\mathcal{L} \supset \int d^2\theta d^2\bar{\theta} \left[I \times K \left(\widehat{B}^\dagger e^{N_c V_B} \widehat{B}, \widehat{\overline{B}}^\dagger e^{-N_c V_B} \widehat{\overline{B}}, (\widehat{B}\widehat{\overline{B}} + \text{h.c.}), \text{tr}\widehat{M}^\dagger \widehat{M}, \ldots \right) \right] \tag{4.8}$$

where we've only written lowest order terms in the fields. Here I is a $\mathrm{U}(1)_A$ and RG-invariant superfield defined in eq. (B.5). The point of eq. (4.8) is that I, which contains the soft mass $\widetilde{m}^2$ as its $\theta^2\bar{\theta}^2$ component, only appears as an overall multiplicative factor, while the background $\mathrm{U}(1)_B$ gauge field which contains the chemical potential μ couples to the baryons as expected. Note that the $\widehat{\overline{B}}\widehat{B}$ term cannot be forbidden by the symmetries, and prevents us from completely determining the baryon soft masses. In later cases such terms will be disallowed by $\mathrm{U}(1)_A$ invariance.

In what follows we will assume that the SUSY-breaking effects are sufficiently small compared to the dynamical scale Λ such that the quantum modified constraint, eq. (4.7), is strictly enforced. We will therefore restrict ourselves to the supersymmetric flat directions while enforcing the constraint by hand. We then minimize the resulting potential including the soft-breaking terms.

Since the origin of moduli space is excluded by the constraint, some of the global symmetries are necessarily broken by the strong dynamics even in the absence of a chemical potential. We will therefore expand the low energy Kähler potential around points that possess the highest degree of remaining symmetry and see whether they remain stable in the presence of a chemical potential. In the following subsections we will analyze the two points of "maximal" symmetry. We cannot determine which point gives the true vacuum of softly-broken supersymmetric gauge theory. However, the first point resembles the dynamics of non-supersymmetric theory the most and hence is a good candidate for the true vacuum. In that case we can show that a large enough chemical potential destabilizes the vacuum, leading to the spontaneous breakdown of $\mathrm{U}(1)_B$.

4.2.1 Vacuum with chiral symmetry breaking

In terms of the hatted fields, a general vacuum that conserves baryon number has $\langle \widehat{B} \rangle = \langle \widehat{\overline{B}} \rangle = 0$ and $\langle \det \widehat{M} \rangle = 1$. Among them, the solution to the constraint $\langle \widehat{M}_{ij} \rangle = \delta_{ij}$ preserves the largest symmetry. At this vacuum of the supersymmetric theory, the $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ chiral symmetry is broken to the diagonal $\mathrm{SU}(N_f)$, while $\mathrm{U}(1)_B$ is unbroken. Therefore this point resembles strongly the dynamics of the non-supersymmetric theory, and hence it is reasonable to expect that the soft supersymmetry breaking effects make this point stable by giving the fluctuations positive squared masses. Indeed, we will show below that this is not unreasonable. Furthermore, since this pattern is similar to that achieved by adding supersymmetric masses it is interesting to compare the results for the two types of masses once a chemical potential is added.

We expand the meson field as $\widehat{M} = \delta_{ij} + \Phi_{ij}$. Following [12], we can now use eq. (4.7) to eliminate one of the degrees of freedom. It is convenient to express the trace of Φ in terms of the other fields,

$$\operatorname{tr}\Phi = \widehat{\overline{B}}\widehat{B} + \frac{1}{2}\operatorname{tr}(\Phi'^2) + \text{higher terms in } \operatorname{tr}\Phi\,, \tag{4.9}$$

where $\Phi'_{ij} = \Phi_{ij} - \delta_{ij}/N_f$ is the traceless component of Φ.

The Kähler potential in eq. (4.8) can now be expressed in terms of the small excitations $\Phi', \widehat{B}$ and $\widehat{\overline{B}}$ and expanded in a power series. For example, substituting into the meson kinetic term in the Kähler potential we get

$$\begin{aligned}\operatorname{tr}\widehat{M}^\dagger\widehat{M} &= N_f + \operatorname{tr}(\Phi'^\dagger\Phi') + \frac{1}{2}\left(\operatorname{tr}(\Phi'^2) + \text{h.c.}\right) + \left(\overline{B}\widehat{B} + \text{h.c.}\right) + \cdots \\ &= N_f + \frac{1}{2}\operatorname{tr}\left(\Phi' + \Phi'^\dagger\right)^2 + \left(\overline{B}\widehat{B} + \text{h.c.}\right) + \cdots\,.\end{aligned} \tag{4.10}$$

The soft masses will ultimately be determined by inserting the SUSY breaking VEVs for V_B and I (Equations (3.2) and (B.6 respectively) and performing the $d^4\theta$ integral in the Kähler potential. The mesons will receive a soft mass only from the SUSY breaking in I. From eq. (4.10) we see that at lowest order in the fields only the real part of Φ' receives a mass while the imaginary part remains massless. However, since we are expanding the meson field around a VEV of order the dynamical scale, masses that come from inserting the VEV into the higher order terms in eq. (4.8) will not be suppressed.

We can use Goldstone's Theorem to show that the imaginary part of Φ' remains massless to all orders in the fields. The Kähler potential of eq. (4.8) can be expanded in powers of the excitations[2]

$$\begin{aligned}\mathcal{L} \supset \int d^2\theta d^2\bar{\theta}\, I \times \Big[c_1\left(\widehat{B}^\dagger e^{N_c V_B}\widehat{B} + \widehat{\overline{B}}^\dagger e^{-N_c V_B}\widehat{\overline{B}}\right) + \left(c_2\widehat{B}\widehat{\overline{B}} + \text{h.c.}\right) + \\ + c_3\operatorname{tr}\Phi'^\dagger\Phi' + c_4\left(\operatorname{tr}\Phi'^2 + \text{h.c.}\right)\Big]\,.\end{aligned} \tag{4.11}$$

The coefficients c_i are derivatives of the function K and include contributions from operators of all orders in the meson field. The coefficients c_1 and c_3 should be absorbed into the definition of the fields in order to have canonically normalized kinetic terms. Performing the superspace integration, the mass matrix for each Φ'_{ij} in the basis $(\Phi', \Phi'^\dagger)$ is

$$m^2_{\Phi'} = \widetilde{m}^2\begin{pmatrix} 1 & c_4/c_3 \\ c_4/c_3 & 1\end{pmatrix}. \tag{4.12}$$

Since the VEV of M breaks the chiral flavor symmetry $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ to the diagonal $\mathrm{SU}(N)_{L+R}$ there must be $N_f^2 - 1$ massless Goldsone Bosons. Since there are $N_f^2 - 1$ Φ' fields this can only work out if all the matrices in eq. (4.12) have a zero eigenvalue. This sets the ratio $c_4/c_3 = 1$ (though -1 will do as well) and the masses are

$$m^2_{\mathrm{Re}\Phi'} = 2\widetilde{m}^2 \qquad m^2_{\mathrm{Im}\Phi'} = 0\,. \tag{4.13}$$

[2]We are assuming that the strong dynamics respect a Z_2 symmetry that interchanges B and $\overline{B}$.

Therefore, the fluctuations in the meson degrees freedom indeed have positive mass-squared for their real parts and zero mass for the Nambu-Goldstone bosons, and hence the vacuum is stable against these fluctuations. Since the mesons are uncharged under baryon number this is unchanged in the presence of a chemical potential.

On the other hand, the situation with the baryonic degrees freedom is less clear. In [12] the meson masses were determined as above and it was noted that since we do not have control over the coefficients of the $\overline{B}B$-type mass, the baryon masses cannot be explicitly determined. Nonetheless, we can still learn about the stability of the potential. After canonically normalizing and performing the θ^4 integral, the baryon mass matrix in the basis $(\widehat{B}\ \widehat{\overline{B}}^\dagger)$ is

$$m_B^2 = \begin{pmatrix} \widetilde{m}^2 - N_c^2\mu^2 & \frac{c_2}{c_1}\widetilde{m}^2 \\ \frac{c_2^*}{c_1}\widetilde{m}^2 & \widetilde{m}^2 - N_c^2\mu^2 \end{pmatrix}. \tag{4.14}$$

The diagonal mass receives a positive contribution from the stabilizing soft mass $\widetilde{m}^2$ and a negative contribution from the chemical potential which is enhanced by a factor of N_c^2. Recall that the requirement for a stable theory is $\widetilde{m}^2 > \mu^2$.

In order to determine whether a chemical potential induces a phase transition we first must check whether baryon number is conserved at $\mu = 0$. In the absence of the chemical potential, the eigenvalues of the mass matrix eq. (4.14) are $(1 \pm |\frac{c_2}{c_1}|)\widetilde{m}^2$. Therefore the vacuum $B = \overline{B} = 0$ is stable if the off-diagonal element is smaller than the diagonal element, $|c_2| < |c_1|$. We cannot rigorously justify that this is the case, though it is a reasonable assumption.[3] Henceforth we assume that this is the case. Then the vacuum is stable against baryonic fluctuations and the dynamics is precisely that of the non-supersymmetric theory: dynamical chiral symmetry breaking with massless Nambu-Goldstone bosons and massive baryons. The main difference, however, is that the baryons are not as heavy as Λ but rather as light as $\widetilde{m} \ll \Lambda$ (within the validity of our approximations).

Now we consider a finite chemical potential. As seen from the mass matrix eq. (4.14), there is a critical value of the chemical potential that makes one of the eigenvalues negative:

$$\mu_c^2 = \frac{1}{N_c^2}\left(1 - \left|\frac{c_2}{c_1}\right|\right)\widetilde{m}^2 . \tag{4.15}$$

Without loss of generality, we can always perform a $\mathrm{U}(1)_B$ rotation to make $c_2/c_1 < 0$. Then the direction of the instability is $\widehat{B} = \widehat{\overline{B}}$. The fields roll down the potential and dynamically break the $\mathrm{U}(1)_B$ symmetry. On the other hand, $|\widehat{B}|, |\widehat{\overline{B}}| \gg 1$ corresponds to the semi-classical regime where

$$Q = \overline{Q} = \begin{pmatrix} v & & \\ & \ddots & \\ & & v \end{pmatrix}, \qquad M = \begin{pmatrix} v^2 & & \\ & \ddots & \\ & & v^2 \end{pmatrix}, \qquad B = \overline{B} = v^{N_c} . \tag{4.16}$$

[3]For $N_c = 2$ the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ flavor symmetry is enlarged to SO(4), which prohibits the $B\overline{B}$ terms. However, the SO(4) is broken by the chemical potential which would be expected to generate a $B\overline{B}$ term with a coefficient of order μ/Λ, which is less than the $\mathcal{O}(1)$ coefficient of the $B^\dagger B$ terms. Thus $|c_2| < |c_1|$ is the natural expectation for $N_c = 2$.

We know the potential is stable in this regime because the dynamics is correctly described in terms of the quark degrees of freedom and $\widetilde{m} > \mu$ assures the stability. Therefore, the baryon fields should stabilize somewhere around $\widehat{B} = \widehat{\overline{B}} \sim 1$. The precise vacuum expectation values depend on the exact form of the Kähler potential and hence cannot be worked out. However the existence of a stable vacuum is obvious from this discussion.

In summary, up to the assumption of $|c_2| < |c_1|$, the supersymmetric vacuum with chiral symmetry breaking and unbroken $\mathrm{U}(1)_B$ is stable in the presence of the soft supersymmetry breaking.[4] There is a critical chemical potential of the order of supersymmetry breaking beyond which $\mathrm{U}(1)_B$ breaks dynamically. The order parameters are expected to all be on the order of the dynamical scale and hence the theory is strongly coupled. Nonetheless the breakdown of $\mathrm{U}(1)_B$ and the presence of a stable vacuum is guaranteed.

This same conclusion holds even if we were to expand around more general baryon number conserving vacua, since the crucial piece of information we need is only the instability of the baryon directions, which follows directly from the diagonal soft mass and chemical potential contributions to the baryon mass terms. Since this is independent of the precise meson VEV, we conclude that (modulo one assumption) *all* baryon conserving vacua are unstable and thus that baryon number is not a global symmetry in the presence of a chemical potential that is larger than μ_c. As in the case with the supersymmetric mass, the critical chemical potential is again of order m/N_c. The breaking pattern also agrees with the result for non-supersymmetric QCD.

4.2.2 Vacuum with baryon number breaking

In the supersymmetric limit, there is another vacuum with a large unbroken symmetry, $\langle \widehat{M} \rangle = 0$ and $\langle \widehat{\overline{B}} \widehat{B} \rangle = -1$. It preserves the full $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ chiral symmetry but breaks baryon number symmetry. We already see that this point is of lesser interest to us since we are focusing on the $\mathrm{U}(1)_B$-breaking signature induced by the chemical potential.

Again we would like to determine whether this point is stable in the presence of soft supersymmetry breaking. The dynamical degrees of freedom around this point are the full meson matrix M and another chiral superfield b: $B = e^b$, $\overline{B} = e^{-b}$. Since $b \to b + i\theta$ under a $\mathrm{U}(1)_B$ transformation, the Kähler potential depends only on $b + b^*$. In the absence of a chemical potential, the Kähler potential for mesons takes the form

$$\mathcal{L} \supset \int d^4\theta I f(b + b^*) \mathrm{Tr} M^\dagger M \,. \tag{4.17}$$

The positivity of the kinetic term for meson fields requires $f(0) > 0$, while the $[\ln I]_{\theta^2\bar{\theta}^2} = -2\frac{N_f}{b_0}\widetilde{m}^2$ gives the mesons a positive mass squared. Therefore, the meson directions are stable around this point.

[4] If $|c_2| > |c_1|$, baryon number is already broken in the absence of a chemical potential. In this case a chemical potential does not change the dynamics qualitatively but it does cause the $\mathrm{U}(1)_B$-breaking to be "stronger". This will cause the critical temperature at which $\mathrm{U}(1)_B$ is restored to grow with μ as one would naively expect (see eq.2.7).

JHEP03(2004)049

Concerning the direction b, the leading term in the Kähler potential is

$$\mathcal{L} \supset \int d^4\theta \frac{1}{2} I (b + b^*)^2 \,, \tag{4.18}$$

which gives mass squared $\widetilde{m}^2$ to the real part while keeping the imaginary part a massless Nambu-Goldstone boson. A linear term $\int d^4\theta I(b + b^*)$ is forbidden by charge conjugation invariance $Q \leftrightarrow \overline{Q}$ that requires the invariance of the effective lagrangian under $b \to -b$. Therefore this point is stable even in the presence of the soft supersymmetry breaking.

Because baryon number is already broken in the absence of the chemical potential, the finite chemical potential can change the physics qualitatively only by breaking the chiral symmetry. However, it is the breaking of baryon number that prevents us from calculating whether this is indeed the case. One possibility is that the meson direction is destabilized by an operator of the type

$$\mathcal{L} \supset \int d\theta^2 d\bar{\theta}^2 \; cI \left(\widehat{B}^\dagger e^{N_c V_B} \widehat{B} \right) \left(\mathrm{tr}\, \widehat{M}^\dagger \widehat{M} \right) \supset c N_c^2 \mu^2 |B_0|^2 \,\mathrm{tr}\, \widehat{M}^\dagger \widehat{M} \tag{4.19}$$

which could give an additional contribution of either sign to the meson mass squared. Thus we cannot determine the fate of the chiral symmetry. In any case, this vacuum is of limited interest to us.

5. $N_f < N_c$

When $N_f < N_c$ the non-supersymmetric results leave an unbroken global U(1). As we will show below this is true in SQCD as well. In this case the IR degrees of freedom consist only of the meson superfield $M = Q\overline{Q}$ since there are not enough quark flavors to form color-singlet baryons. The SQCD dynamics generates the so-called Affleck-Dine-Seiberg superpotential,

$$W = (N_c - N_f) \frac{\Lambda^{3+2N_f/(N_c - N_f)}}{(\det M)^{1/(N_c - N_f)}} \,. \tag{5.1}$$

Because of this potential, the meson field is driven away from the origin, leading to a run-away theory with no vacuum.

However, this runaway behavior is stopped by the masses we add in the UV. Let us begin with a supersymmetric mass and assume that this mass is large, $m > \Lambda$. Adding this mass to the superpotential and solving for the meson VEV we get the same formula as in the previous section

$$\langle M_{ij} \rangle = (m^{-1})_{ij} \left(\Lambda^{b_0} \det m \right)^{1 - N_f/N_c} \sim \left(\frac{\Lambda}{m} \right)^{1/N_c} \Lambda^2 \,. \tag{5.2}$$

Note that the VEV is smaller than Λ and therefore the meson is indeed the appropriate degree of freedom to describe the IR vacuum. We can immediately see that baryon number is unbroken because the IR degrees of freedom are not charged under $\mathrm{U}(1)_B$. For this case adding a chemical potential has no effect.

JHEP03(2004)049

We will now consider adding small masses $m < \Lambda$. In this case the supersymmetric masses and the soft masses yield similar results so we will treat them together. When the stabilizing mass is small the mesons runs far away from the origin before it is stabilized at VEVs much larger than Λ. This region of moduli space is described in terms of quarks. The large meson amplitude corresponds to the flat direction

$$Q = \overline{Q} = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_{N_f} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} . \tag{5.3}$$

Rewriting the ADS potential in terms of quarks and adding soft or supersymmetric masses and the tachyonic mass due to the chemical potential we can minimize and solve for $v_1, \ldots, v_{N_F}$. For example, when soft masses are added such that $0 < \widetilde{m}^2 - \mu^2$, we get

$$v_1 = \cdots = v_{N_f} \sim \Lambda \left(\frac{\Lambda}{\sqrt{\widetilde{m}^2 - \mu^2}} \right)^{(N_c - N_f)/2N_c} . \tag{5.4}$$

The form of the VEV is slightly different when a supersymmetric mass is used, but $v_1 = \cdots = v_{N_f}$ is still satisfied. Therefore this theory dynamically breaks the chiral symmetry $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \to \mathrm{SU}(N_f)$. However, a $\mathrm{U}(1)_{\widetilde{B}}$ remains unbroken. The new baryon symmetry is the simultaneous transformation under the original $\mathrm{U}(1)_B$ and a $\mathrm{U}(1)'$, where $\mathrm{U}(1)'$ is a subgroup of the gauge symmetry embedded as $\mathrm{SU}(N_c) \supset \mathrm{SU}(N_f) \times \mathrm{SU}(N_c - N_f) \times \mathrm{U}(1)'$. This situation is analogous to the non-supersymmetric case of $N_f = 2$ and $N_c = 3$ (the so-called 2SC phase) where the unbroken U(1) is a combination of baryon number and a broken color generator.

6. $N_f = N_c + 1$

When $N_f = N_c + 1$ the IR theory is again described in terms of mesons $M_{ij} = Q_i^a \overline{Q}_j^a$ and baryons $B_i = \epsilon_{ij_1 \cdots j_{N_c}} Q_{j_1} \cdots Q_{j_{N_c}}$ and $\overline{B}_i = \epsilon_{ij_1 \cdots j_{N_c}} \overline{Q}_{j_1} \cdots \overline{Q}_{j_{N_c}}$, but the symmetry transformation properties are different. Choosing the quarks to have baryon number 1, the fields M, B and $\overline{B}$ transform as $(N_f, \overline{N}_f)_0$, $(\overline{N}_f, 1)_{N_c}$ and $(1, N_f)_{-N_c}$ respectively, under the global symmetry $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_B$.

We summarize our results before presenting the details. Recall that for $N_f = 4$ in non-supersymmetric QCD baryon number is broken in the presence of a high chemical potential. For SQCD, adding a supersymmetric mass in a specified range will again lead to a critical chemical potential above which baryon number is broken, in agreement with the QCD result. Adding UV soft masses yields only partial agreement. If we choose to stay along a flat direction that breaks the chiral symmetry to a vector symmetry we indeed find that the existence of a critical chemical potential is plausible. However, there is also another flat direction along which the chiral symmetry is unbroken which has tachyonic masses for the baryons, thus breaking $\mathrm{U}(1)_B$ even for $\mu = 0$.

6.1 Supersymmetric mass

As in the previous case we will stabilize the UV theory first with supersymmetric masses m and then with soft SUSY-breaking masses $\widetilde{m}$.

The low energy superpotential for baryons and mesons, including the supersymmetric quark masses, is

$$W = \frac{1}{\Lambda^{b_0}}(\mathrm{det}M - B_i M_{ij} \overline{B}_j) + \mathrm{Tr}(mM)\,. \tag{6.1}$$

The baryon VEVs vanish but the mesons acquire a nonzero VEV:

$$\langle M_{ij} \rangle = (m^{-1})_{ij} \left(\Lambda^{b_0} \det m\right)^{1/N_c}\,. \tag{6.2}$$

The meson VEV yields a mass for the baryons $\int d^2\theta(\langle M \rangle / \Lambda^{b_0})\overline{B}B$. Including the chemical potential contribution, the mass-squared for the canonically normalized baryon and antibaryon fields is

$$m^2_{B,\overline{B}} = \frac{1}{c_0}\left(\frac{m}{\Lambda}\right)^{2/N_c} \Lambda^2 - N_c^2 \mu^2 \tag{6.3}$$

where again c_0 is an unknown $\mathcal{O}(1)$ coefficient in the Kähler potential, and we have taken the supersymmetric mass $m_{ij} = m\delta_{ij}$ in order to preserve as much global symmetry as possible.

Here again there is a critical chemical potential μ_c above which the baryons become unstable and break $\mathrm{U}(1)_B$:

$$\mu_c^2 = \frac{1}{c_0 N_c^2}\left(\frac{m}{\Lambda}\right)^{2/N_c} \Lambda^2\,. \tag{6.4}$$

However, this formula will only be valid as long as $\mu_c^2 < \Lambda^2$ (near-SUSY limit) and $\mu_c^2 < m^2$ (UV stability). These two constraints yield the requirements

$$(c_0 N_c^2)^{N_c/(2-2N_c)} < \frac{m}{\Lambda} < (c_0 N_c^2)^{N_c/2}\,. \tag{6.5}$$

These can be satisfied as long as $c_0 N_c^2 > 1$, since (for $N_c > 1$) the lower bound is a decreasing function of $c_0 N_c^2$ and the upper bound is an increasing function. This is the same restriction on c_0 that we found in the case where $N_f = N_c$. We again conclude that it is very likely that baryon number is broken by a sufficiently large chemical potential. There are, however, some differences compared to the $N_f = N_c$ result. The critical chemical potential is not $\mathcal{O}(m)$ but is a combination of m and Λ. Also note that the UV mass is bounded from below in order for our result to hold, as shown in the right diagram of figure 1.

6.2 SUSY-breaking mass

Now we consider a different situation where m is set equal to zero in eq. (6.1) and instead we add universal soft SUSY-breaking masses $\widetilde{m}$ for the squarks in the UV theory. Using the results of [10] reviewed in appendix B, the soft-masses for the squarks result in the following soft masses for the mesons and baryons:

$$\widetilde{m}^2_M = \frac{2N_c - 4}{2N_c - 1}\widetilde{m}^2 \qquad \text{and} \qquad \widetilde{m}^2_{B,\overline{B}} = \frac{2 - N_c}{2N_c - 1}\widetilde{m}^2 - N_c^2 \mu^2\,. \tag{6.6}$$

JHEP03(2004)049

Notice that for $N_c \geq 3$ the meson mass-squared is positive, but the baryons are tachyonic. Here we have also included the contribution from the chemical potential, which simply follows from baryon number.

When we canonically normalize the baryon and meson superfields the unknown constants $c_{M,B,\overline{B}}$ reappear as relative coefficients between the terms of the superpotential:

$$W = c_1 \frac{\det M'}{\Lambda_h^{N_f-3}} - c_2 B'_i M'_{ij} \overline{B}'_j \tag{6.7}$$

where $M' = M/\Lambda_h$ and $B' = B/\Lambda_h^{N_c-1}$ are dimension one fields, and $c_1 = c_M^{-N_f/2}$ and $c_2 = (c_M c_B c_{\overline{B}})^{-1/2}$. This, along with the soft masses of eq. (6.6), gives rise to the following scalar potential

$$\begin{aligned} V_{IR} = & \left| c_1 \frac{\widetilde{m}'_{ij}}{\Lambda_h^{N_f-3}} - c_2 B'_i \overline{B}'_j \right|^2 + \left| c_2 M'_{ij} \overline{B}'_j \right|^2 + \left| c_2 B'_i M'_{ij} \right|^2 + \\ & + \sum_{ij} \widetilde{m}_M^2 \left| M'_{ij} \right|^2 + \sum_i \widetilde{m}_B^2 \left| B'_i \right|^2 + \sum_i \widetilde{m}_{\overline{B}}^2 \left| \overline{B}'_i \right|^2 \end{aligned} \tag{6.8}$$

where $\widetilde{m}'_{ij}$ is the ijth cofactor of M'_{ij}. In the supersymmetric limit this potential has two flat direction with different symmetry breaking patterns. We will now analyze these flat directions separately.

The first flat direction has VEVs of the form:

$$B = \overline{B} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix}. \tag{6.9}$$

Along this flat direction the relationship between the VEVs is constrained to be $c_1 a^{N_c} = c_2 \Lambda^{N_c-2} b^2$ by the first term in eq. (6.8). The introduction of the soft masses has the potential to destabilize this flat direction. The potential along the flat direction for $|a| < \Lambda$ is given by

$$V_{IR} = N_c \frac{2N_c - 4}{2N_c - 1} \widetilde{m}^2 |a|^2 - 2 \left(\frac{N_c - 2}{2N_c - 1} \widetilde{m}^2 + N_c^2 \mu^2 \right) \left| \frac{c_1}{c_2} \frac{a^{N_c}}{\Lambda^{N_c-2}} \right|. \tag{6.10}$$

On the other hand, at large amplitudes the quark description is valid. The flat direction above corresponds to the quark VEVs

$$Q = \overline{Q} = \left(\begin{array}{cccc|c} 0 & & & & 0 \\ & \sqrt{a} & & & 0 \\ & & \ddots & & \vdots \\ & & & \sqrt{a} & 0 \end{array} \right) \tag{6.11}$$

and hence, for $|a| > \Lambda$,

$$V_{IR} = 2N_c(\widetilde{m}^2 - \mu^2)|a| \,. \tag{6.12}$$

Clearly $V_{IR} = 0$ at the origin. The potential goes up as $|a|^2$ first, but then goes down as $-|a|^{N_c}$ at a larger amplitude. We will have a vacuum with energy lower than the origin if the potential in eq. (6.10) becomes negative within its range of applicability, namely $|a| < \Lambda$. Beyond this region the potential goes up linearly with $|a|$ as in eq. (6.12).

When $\mu = 0$, V_{IR} will be negative when

$$\left(N_c \left|\frac{c_2}{c_1}\right|\right)^{1/(N_c-2)} < \frac{|a|}{\Lambda} < 1 \tag{6.13}$$

where the upper-bound is the requirement for eq. (6.10) to be valid. This condition is somewhat restrictive, though by no means impossible to satisfy. For $N_c = 3$ it requires $|c_1| > N_c|c_2|$ with $|a|$ near its upper bound.

Nonzero μ helps make V_{IR} more negative. In particular, there is a critical chemical potential given by

$$\mu_c^2 = \frac{N_c - 2}{N_c^2(2N_c - 1)} \left[N_c \left|\frac{c_2}{c_1}\right| \left(\frac{\Lambda}{|a|}\right)^{N_c-2} - 1\right] \widetilde{m}^2 \tag{6.14}$$

above which V_{IR} goes negative, as long as

$$\left(\left|\frac{c_2}{c_1}\right| \left[\frac{N_c(N_c-2)}{N_c^2(2N_c-1) + (N_c-2)}\right]\right)^{1/(N_c-2)} < \frac{|a|}{\Lambda} < 1\,. \tag{6.15}$$

The lower bound comes from requiring $\mu < \widetilde{m}$, whereas the upper bound is again required for eq. (6.10) to be valid. The extra factors of N_c on the left-hand side make the requirements on c_1 and c_2 less stringent. However, as N_c grows the allowed range for $|a|$ decreases. If indeed such a point is the global minimum, baryon number will be broken, and the chiral flavor symmetries will be broken to the diagonal $\mathrm{SU}(N_f - 1)_V$ vector symmetry once μ exceeds μ_c.

We will now analyze the second flat direction. When $M = 0$ a SUSY vacuum requires at least one of B_i or $\overline{B}_i$ to also vanish. For example, take $B_i = 0$ for all i. The positive meson mass-squared will tend to keep the meson VEV zero, but the negative baryon mass-squared will draw the $\overline{B}$ VEVs away from the origin. Specifically, flavor symmetry allows us to write the VEVs as

$$B = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{and} \qquad \overline{B} = \begin{pmatrix} \bar{b} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6.16}$$

which will be pulled in the direction $|b|^2 \neq 0, |\bar{b}|^2 = 0$ or vice versa. In the former case the flavor symmetry is broken to $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$. Once the amplitude is larger than Λ, this direction corresponds to

$$Q = \left(\begin{array}{cccc|c} 0 & & & & 0 \\ & b^{1/N_c} & & & 0 \\ & & \ddots & & 0 \\ & & & b^{1/N_c} & 0 \end{array}\right), \qquad \overline{Q} = 0\,. \tag{6.17}$$

Again the potential goes up as $V_{IR} = N_c(\widetilde{m}^2 - \mu^2)|b|^{2/N_c}$ at large amplitudes. Thus the vacuum of the theory will settle at an intermediate scale where we lose calculability.

Since we did not truly minimize the potential, the two symmetry breaking patterns above should be taken as candidates. One may hope that once the theory begins to roll down one of these directions it will 'try to maintain as much of its symmetry' and stay along that direction. However, it is possible that the true minimum lies elsewhere.

In summary, when $N_f = N_c + 1$ and soft masses are added, the origin of moduli space is unstable and thus some of the SQCD global symmetries must be broken. We found two candidate directions. In the first case the theory is not unstable right at the origin but becomes so near the origin as long as N_c is not too big. In this direction the chiral symmetries are broken to the vector subgroup, and the meson VEVs as in eq. (6.9) show a color-flavor locked pattern for N_c of the flavors. However, in the second direction, which is unstable for any number of colors, some of the chiral symmetries remain unbroken. It appears that there is a first order phase transition between the two directions as μ is increased.

In both cases baryon number is broken dynamically, consistent with the result in non-supersymmetric QCD. Both of the candidate symmetry breaking patterns differ from the previous results for non-supersymmetric QCD as shown in table 1.

7. $N_c + 2 \leq N_f < 3N_c$ — dual region

In this section we would like to analyze SQCD when $N_c + 2 \leq N_f < 3N_c$. Seiberg has shown [14] that the original SU(N_c) "electric" theory is dual to a different "magnetic" gauge theory. The dual theory consists of an SU($\widetilde{N}_c$) gauge theory, where $\widetilde{N}_c = N_f - N_c$, with N_f flavors of magnetic quarks q_i and $\bar{q}_i$ along with a gauge singlet M_{ij} and superpotential $W = \frac{1}{k} q_i M_{ij} \bar{q}_j$.[5] The two theories are obviously different in the UV, but they are dual in the sense that they describe the same IR physics.

When $N_c + 2 \leq N_f < \frac{3}{2}N_c$, the so-called free magnetic phase, the dual description is particularly useful because it is IR free. In that case, the theory of dual quarks and the meson is calculable in the IR and our goal is clear: we want to determine whether the mass squared of the dual quarks is negative when a chemical potential is added.

When $\frac{3}{2}N_c < N_f < 3N_c$ both the electric and the magnetic theories are asymptotically free but flow to an IR fixed point. Our analysis hinges on the ability to identify the correct low energy degrees of freedom, their masses, and their charge under U$(1)_B$. Once these are specified the effect of the chemical potential is determined by gauge invariance and we can determine the stability of the origin. However, when the IR physics is strongly coupled the quarks (electric or magnetic) may no longer be the correct degrees of freedom. Therefore the fixed point value of the gauge coupling is crucial in determining the reliability of our result. As N_f is increased within this range the fixed point coupling for the electric theory decreases whereas the fixed point coupling of the dual magnetic theory increases. Once $N_f > 3N_c$ the electric theory becomes IR-free and the magnetic theory is asymptotically free. Therefore, when $\frac{3}{2}N_c < N_f < 2N_c$ the magnetic theory is more reliable, whereas for $2N_c < N_f < 3N_c$ it is the electric theory we should be using. Near $N_f = 2N_c$ both theories are strongly coupled, so our analysis breaks down in that region. Unfortunately

[5]Here we use k instead of the conventional μ in order to distinguish it from the chemical potential.

JHEP03(2004)049

we cannot determine the fixed point couplings beyond the qualitative discussion above. Since all we can do is calculate the mass squared of either the electric or magnetic quarks we will therefore trust the result only in the two edges of the conformal window where one of the descriptions is known to be weakly coupled.

Before we discuss our results we will further comment on calculating the critical chemical potential in the conformal window. Below we will use exact supersymmetric results to calculate the masses of the IR degrees of freedom, either electric or magnetic. The mass calculated in this manner is the effective mass at the scale Λ, where the theory becomes conformal. If one would like to calculate the masses at a lower scale ω, the aproach to the fixed point must be considered. The wavefunction renormalization of the quarks is given by

$$Z = \left(\frac{\omega}{\Lambda}\right)^{\gamma_*} \tag{7.1}$$

where $\gamma_* = b_0/N_f$ (or $\tilde{b}_0/N_f$ in the magnetic case) is determined by the anomaly free R-charge. This decrease in Z leads to an *enhancement* in the effective IR masses $m_{phys} = m/Z$. One may worry that the increase in the stabilizing mass might make the breaking of $\mathrm{U}(1)_B$ by a chemical potential more difficult. However, this is not the case. We may write the effective potential for the squrk at the scale $\omega \sim q$ as

$$V_{\text{eff}}(q) = m^2(q)|q|^2\,. \tag{7.2}$$

If the effective mass $m^2(q)$ is negative for some scale (not necesarily the IR) the potential becomes negative there and a non-trivial minimum exists. Since the supersymmetric mass is only enhanced below Λ, our best chance of achieving an instability is for a scale of order Λ.

The summary of our result for this case is as follows. When we add supersymmetric masses and focus on the free magnetic phase we find that a large enough chemical potential will make the dual quark masses negative and break $\mathrm{U}(1)_B$, provided a constraint on k/Λ is satisfied. Notice that this phase does not exist for $N_c = 3$ so a comparison with QCD results cannot be made.

For $N_f = \frac{3}{2}N_c$ there is no critical chemical potential within our approximations. For $\frac{3}{2}N_c < N_f < 2N_c$ a critical chemical potential may exist if a certain restriction on k/Λ is satisfied. However the naive expectation of $k \sim \Lambda$ does not satisfy the restriction. This is to be compared with the QCD result that $\mathrm{U}(1)_B$ is broken for $N_f = 5$ which is at the bottom of the conformal window ($N_f = 1.66N_c$).

If we extrapolate our results to $N_f = 2N_c$ we find that the critical chemical potential needed to destabilize the magnetic quarks approaches the electric quark mass. This shows that the theory maintains its self duality for $N_f = 2N_c$ when a chemical potential is added, and might indicate that $\mathrm{U}(1)_B$ can no longer be broken dynamically in either description. However, this is the region where we have the least control since both descriptions of the theory are strongly coupled.

Extrapolating further to $2N_c < N_f < 3N_c$ we again find that the mass squared for the magnetic squarks may become negative with a sufficient chemical potential (without requirements on k), however in this region the electric quarks are more reliable. By our construction the electric quarks do not condense dynamically and therefore $\mathrm{U}(1)_B$ is not broken for this case either.

JHEP03(2004)049

When soft-masses are added in the UV theory we have control over the IR masses only in the free magnetic phase. There baryon-number is broken even without a chemical potential. Even though a finite chemical potential leads to a "stronger" breaking of baryon number, it does not lead to a phase transition like that found in other cases.

In the conformal window, in addition to the difficulties described above, the IR soft masses are uncalculable. Qualitatively we expect that near $N_f = \frac{3}{2}N_c$ baryon number is dynamically broken by a chemical potential and near $N_f = 3N_c$ it is not. This will be argued in section 7.2.2.

7.1 Supersymmetric mass

First we will consider adding a supersymmetric mass for all the quarks in the original theory in the form of a superpotential $W_{\rm mass} = m_{ij}Q_i\overline{Q}_j$, with $m_{ij} = m\delta_{ij}$ as before, which preserves the vector flavor symmetry. In the dual theory, we thus have a superpotential

$$W_{\rm dual} = \frac{1}{k}q_i M_{ij}\bar{q}_j + {\rm Tr}(mM)\,. \tag{7.3}$$

The equations of motion for M yield $q_i\bar{q}_j = m_{ji}$, which does not admit any solutions because $q_i\bar{q}_j$ is at most a rank $\widetilde{N}_c$ matrix, whereas m_{ij} is chosen to have rank $N_f > \widetilde{N}_c$. However, since we know that $\langle M\rangle \neq 0$ in the electric theory, it suggests that the same must be true in the magnetic theory. Then $\langle M\rangle$ becomes a mass term for the dual quarks q, so we can integrate them out, leaving a pure SU$(\widetilde{N}_c)$ Yang-Mills theory with an additional gauge singlet M. The Yang-Mills theory undergoes gaugino condensation, which gives a superpotential term $W_{\lambda\lambda} = \widetilde{N}_c\tilde{\Lambda}_{IR}^3$. The usual matching conditions give $\tilde{\Lambda}_{IR}^{\tilde{b}_0'} = \tilde{\Lambda}^{\tilde{b}_0}\det\langle M\rangle$, so together with the superpotential for M we find an effective superpotential

$$W_{\rm eff} = \widetilde{N}_c\left(\tilde{\Lambda}^{3\widetilde{N}_c - N_f}\det\langle M\rangle\right)^{1/\widetilde{N}_c} + {\rm Tr}(mM)\,. \tag{7.4}$$

Minimizing with respect to M_{ij} gives an expression for m_{ij} which, together with the duality relation $\Lambda^{b_0}\tilde{\Lambda}^{\tilde{b}_0} = (-1)^{\widetilde{N}_c}k^{N_f}$, can be inverted to give $\langle M_{ij}\rangle = (m^{-1})_{ij}\left(\Lambda^{b_0}\det m\right)^{1/N_c}$, which is consistent with the assumption that $\langle M\rangle \neq 0$ and matches the result in the electric theory.

7.1.1 Free magnetic phase

In the free magnetic phase we can estimate the size of k as follows. In the duality relation, Λ is the scale where we expect the electric theory to become strongly coupled, whereas $\tilde{\Lambda}$ is the high scale where the IR-free magnetic theory is strong. Thus it is reasonable to identify these two scales, so that the theory is described by the UV-free electric theory that flows down and becomes strong around $\Lambda = \tilde{\Lambda}$ where the IR-free magnetic theory takes over at strong coupling and subsequently flows to weak coupling in the IR. With this identification, k is also naturally the same size as Λ. This argument also holds for the lower edge of the conformal window where the magnetic theory is weak but for a larger number of flavors the size of k is unknown.

When we add a chemical potential the dual quarks will get a negative mass-squared proportional to their baryon number, namely $N_c/\widetilde{N}_c$, in addition to the mass from $\langle M\rangle$. So altogether,[6]

$$m_{q,\bar{q}}^2 = \frac{1}{k^2}\langle M\rangle^2 - \left(\frac{N_c}{\widetilde{N}_c}\right)^2 \mu^2 = \frac{1}{k^2}\Lambda^{2b/N_c} m^{2(N_f/N_c-1)} - \left(\frac{N_c}{\widetilde{N}_c}\right)^2 \mu^2, \tag{7.5}$$

which leads to a critical chemical potential

$$\mu_c^2 = \left(\frac{\widetilde{N}_c}{N_c}\right)^2 \left(\frac{\Lambda}{k}\right)^2 \left(\frac{m}{\Lambda}\right)^{2(N_f/N_c-1)} \Lambda^2 \,. \tag{7.6}$$

Thus for $\mu > \mu_c$ the dual squarks will have tachyonic masses, signaling an instability. Because all the squarks get the same mass, we expect them to all get the same VEV x which, in order to satisfy the $\mathrm{SU}(\widetilde{N}_c)$ D-term equations, is of the following form in terms of $\widetilde{N}_c \times N_f$ matrices:

$$q = \begin{pmatrix} x & & & \\ & x & & \\ & & x & \\ & & & \ddots \end{pmatrix} \qquad \text{and} \qquad \bar{q} = \begin{pmatrix} \bar{x} & & & \\ & \bar{x} & & \\ & & \bar{x} & \\ & & & \ddots \end{pmatrix}. \tag{7.7}$$

Such a VEV completely breaks the $\mathrm{SU}(\widetilde{N}_c)$ gauge group, which means there is no residual U(1) that could be combined with $\mathrm{U}(1)_B$ to give an unbroken U(1), hence we conclude that μ exceeding the critical chemical potential will break baryon number. But we still need to check that such a value of μ is permitted within the validity of our approximations.

We require $\mu_c < \Lambda$ so that the strong-coupling results can be applied, which leads to an upper bound on m/Λ,

$$\frac{m}{\Lambda} < \left(\frac{N_c}{\widetilde{N}_c}\frac{k}{\Lambda}\right)^{N_c/(N_f-N_c)} . \tag{7.8}$$

The requirement that $\mu_c < m$ for stability in the UV leads to another constraint:

$$\frac{m}{\Lambda} > \left(\frac{N_c}{\widetilde{N}_c}\frac{k}{\Lambda}\right)^{N_c/(2N_c-N_f)} . \tag{7.9}$$

The magnetic theory is free in the infrared, so we are justified in using the description in terms of dual quarks. Furthermore, the upper bound on m/Λ has a larger exponent than the lower bound this time, which means we need $k/\Lambda > N_f/N_c - 1$, which is a quite reasonable constraint since the right hand side is always less than 1/2. As argued above, we expect k and Λ to be comparable. This means that it is reasonable to expect $k/\Lambda > N_f/N_c - 1$, yielding a viable window where $\mu > \mu_c$ exists and all our approximations remain valid. Therefore there is a critical chemical potential, of the form shown in the right diagram of figure 1, above which baryon number is broken.

[6]This formula assumes that the Kähler potential $\int d^4\theta Z_q q^* q$ for the dual quarks is canonical ($Z_q = 1$). If not, the normalization Z_q appears in the formula by replacing k by $Z_q k$. In fact, it is known that scales k, Λ, $\tilde{\Lambda}$ are not physical, but rather depend on the normalization in Kähler potential [11]. This formula as well as the intuition $k \simeq \Lambda \simeq \tilde{\Lambda}$ make sense in the *convention* that the Kähler potential is canonical both for the electric quark in the UV limit and the magnetic quark in the IR limit.

7.1.2 The conformal window

In this case we obtain the same formula for the dual quark mass as in eq. (7.5). The validity requirement of $\mu_c < \Lambda$ also leads to the same constraint eq. (7.8). On the other hand, the requirement that $\mu_c < m$ for stability in the UV leads to three separate cases depending on the size of N_f:

$$\frac{m}{\Lambda} > \left(\frac{N_c}{\widetilde{N}_c}\frac{k}{\Lambda}\right)^{N_c/(2N_c-N_f)} \qquad \text{for } N_f < 2N_c \tag{7.10}$$

$$\frac{m}{\Lambda} < \left(\frac{N_c}{\widetilde{N}_c}\frac{k}{\Lambda}\right)^{N_c/(N_f-2N_c)} \qquad \text{for } N_f > 2N_c \tag{7.11}$$

$$\mu_c = \frac{\Lambda}{k}m \qquad \text{for } N_f = 2N_c\,. \tag{7.12}$$

We see that there are three situations. For $N_f = 2N_c$, μ_c is directly proportional to m so we need $\Lambda/k < 1$. For more flavors, $2N_c < N_f < 3N_c$, there are two different upper bounds, so it is easy to choose a small m that satisfies both. However, this region is the upper part of the conformal window where the magnetic theory is more strongly coupled, so our analysis in terms of the magnetic quarks is unlikely to be valid. Instead it is more appropriate to consider the electric theory where $\mathrm{U}(1)_B$ is not dynamically broken by construction, since we require $\mu < m$.

The more interesting situation is when $N_f < 2N_c$. In this case m/Λ is bounded both from above and from below. For $N_f = \frac{3}{2}N_c$ the two bounds coincide, yielding $\mu_c = \Lambda$, which doesn't leave any room for $\mu > \mu_c$ within our approximations. For $\frac{3}{2}N_c < N_f < 2N_c$ the exponent of the lower bound is larger than the exponent of the upper bound, so for there to be any space between them, we need $(N_c/\widetilde{N}_c)(k/\Lambda) < 1$, which means $k/\Lambda < N_f/N_c - 1$, which is always less than 1 for this range of flavors. While not impossible to satisfy, the naive expectation for N_f close to $\frac{3}{2}N_c$ is that $k/\Lambda \sim 1$.

We conclude that $\mathrm{U}(1)_B$ remains unbroken in the conformal window.

7.2 SUSY-breaking mass

We now wish to add soft masses for the electric squarks in the UV. Again we discuss the free magnetic phase and the conformal window separately.

7.2.1 Free magnetic phase

Since the soft mass modifies the electric theory we must modify the magnetic theory as well in order for the two theories to describe the same moduli space. In [10] it was shown by arguments similar to those of appendix B that the duality is maintained if soft masses are added to the magnetic quarks and the meson. In the deep IR these soft masses take the form

$$m_M^2 = 2\frac{3N_c - 2N_f}{3N_c - N_f}\widetilde{m}^2 \qquad \text{and} \qquad m_{q,\bar{q}}^2 = -\frac{3N_c - 2N_f}{3N_c - N_f}\widetilde{m}^2 - \left(\frac{N_c}{\widetilde{N}_c}\right)^2 \mu^2\,. \tag{7.13}$$

where we have included the contribution due to the chemical potential. Note that the meson soft masses are positive, but as in the previous case, the dual quarks q and $\bar{q}$ get tachyonic masses. Again, since the magnetic theory is free in the IR, its quarks and meson are the appropriate degrees of freedom to analyze near the origin of moduli space.

To analyze the symmetry breaking pattern first note that since the F-term potential derived from eq. (7.3) does not relate the expectation value of the dual quarks to the meson (or its cofactor), the meson VEV will always vanish. Representing the squark and anti-squark as two $\widetilde{N}_c \times N_f$ matrices, the D-term equations can be satisfied with a VEV of the form

$$q = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & \ddots \end{pmatrix} \qquad \text{and} \qquad \bar{q} = \begin{pmatrix} \bar{x}_1 & & & \\ & \bar{x}_2 & & \\ & & \bar{x}_3 & \\ & & & \ddots \end{pmatrix} \tag{7.14}$$

with $|x_i|^2 - |\bar{x}_i|^2 = r$ where r is a common constant for all i.

In order to find the vacuum we must minimize the rest of the potential (the non D-terms) with respect to x_i and the constant r. We can assume without loss of generality that $r > 0$. The potential including the soft terms is

$$V = \sum_i |c_2|^2 |x_i|^2 \left(|x_i|^2 + r\right) + \sum_i m_q^2 \left(2|x_i|^2 + r\right) \tag{7.15}$$

where m_q^2 is negative as in eq. (7.13). This is minimized for $x_i = 0$ with r going to infinity due to the negative mass squared. Since we have stabilized the theory for VEVs far away from the origin we expect r to stabilize at some finite value of order Λ. The instability of the origin exists even in the absence of a chemical potential since the dual squark masses in eq. (7.13) are negative. The case of $N_f = \frac{3}{2} N_c$ is an exception since then the masses of eq. (7.13) vanish when μ is set to zero, and r is a modulus. In that case an arbitrarily small chemical potential will destabilize the origin and break $\mathrm{U}(1)_B$.

Again we can only exclude the origin as a possible vacuum and point in the direction the theory will roll away from the origin. If the vacuum indeed lies along this direction the symmetry breaking pattern resembles the second flat direction for $N_f = N_c + 1$ with soft SUSY-breaking masses. Referring back to the quark VEVs in eq. (7.14) we see that this vacuum preserves the symmetry $\mathrm{SU}(\widetilde{N}_c)_{R+c} \times \mathrm{SU}(N_f - \widetilde{N}_c)_R \times \mathrm{SU}(N_f)_L = \mathrm{SU}(\widetilde{N}_c)_{R+c} \times \mathrm{SU}(N_c)_R \times \mathrm{SU}(N_f)_L$.[7] Negative r would correspond to the situation with left and right interchanged. We see here that the free magnetic phase exhibits some color-flavor locking in the sense that the (dual) color gauge group and a subgroup of the flavor symmetry are broken to the diagonal.

7.2.2 The conformal window

In [12] it was shown that, unlike the supersymmetric masses that get enhanced, the soft masses in both the electric and the dual magnetic theories flow to zero in the IR. The rate at which this occurs is unknown and therefore we cannot determine whether the theory reaches the fixed point. Let us illustrate this qualitatively by focusing on the electric description. When the theory is weakly coupled (near the Banks-Zaks fixed point [15]) the soft mass flows toward zero very slowly so we can approximate it to be constant at its bare value $\widetilde{m}_0$. The conformal symmetry will be broken at the scale of the soft

[7] Note that the $\mathrm{SU}(N_c)$ symmetry here is a *flavor* symmetry.

mass. When the theory is more strongly coupled the soft mass $\widetilde{m}(\omega)$ is a more rapidly decreasing function of the scale ω. In this case the conformal symmetry will be broken at a scale that is equal to the soft mass at that scale $\omega \sim \widetilde{m}(\omega)$. In this case the IR theory will not be conformal nor supersymmetric, though the SUSY- breaking operator has been suppressed.

Finally we may imagine that at yet stronger coupling the soft mass $\widetilde{m}(\omega)$ flows down with scale faster than the scale ω itself. In this case the condition $\omega \sim \widetilde{m}(\omega)$ is never satisfied and the theory reaches a truly conformal fixed point. Note that even though supersymmetry was broken in the UV, it is restored in the IR. (See [16] for an application of such a scenario as a solution for the hierarchy problem.) However, as was pointed out in [12], since we cannot calculate the rate of approach to the fixed point we cannot determine whether this occurs.

From the qualitative picture just described we can get a rough idea about the size of the critical chemical potential for various numbers of flavors within the conformal window. Near $N_f = \frac{3}{2}N_c$ when the electric theory is strongest the IR soft mass is suppressed the most and a smaller chemical potential is needed to break $\mathrm{U}(1)_B$. Since the suppressed mass is obviously smaller than the UV mass, our assumption $\mu < \widetilde{m}_0$ is satisfied and the U(1) breaking is indeed dynamical. If the last scenario of the previous paragraph occurs, baryon number is broken for an arbitrarily small μ.

As we add flavors and move up in the conformal window the suppression weakens and the critical chemical potential rises. The naive expectation is that it will reach $\widetilde{m}_0$ near $N_f = 3N_c$. Note that this matches the values of the critical chemical potential obtained outside of the conformal window: $\mu_c = 0$ for $N_f = \frac{3}{2}N_c$ in section 7.2.1, and $\mu_c = m$ for $N_f \geq 3N_c$ where the theory is IR free, by the arguments of section 2. The rising trend for the critical chemical potential is also consistent with the results for supersymmetric stabilizing masses in section 7.1.

8. Conclusions

Using known exact results for supersymmetric SU(N) gauge theories and soft masses, we have studied the effect of a baryon chemical potential $\mu < \Lambda$ on the global symmetries of the ground state, in particular $\mathrm{U}(1)_B$. Our results are summarized in table 2.

We have found that for $N_f < N_c$ a modified $\mathrm{U}(1)_B$ remains unbroken at low energy, whereas for $N_c \leq N_f < \frac{3}{2}N_c$ there is a critical chemical potential above which baryon number symmetry is broken due to a combination of strong gauge dynamics and the chemical potential. Notice that there is reasonable agreement between the SUSY-preserving and SUSY-breaking masses. Comparing with table 1, we find that $\mathrm{U}(1)_B$ suffers the same fate as in non-supersymmetric QCD ($N_c = 3$) where it is unbroken in the 2SC phase ($N_f = 2$) but broken in the CFL phase ($N_f = 3$). For SQCD with $N_f > 3$ we also find interesting phases that include possible $\mathrm{U}(1)_B$ breaking. For $N_f = 4$ we again find $\mathrm{U}(1)_B$ breaking, in agreement with the non-supersymmetric case. Our results lend support to the results obtained using the variational approach, even though the latter are derived in a very different limit, namely $\mu \gg \Lambda$, and using very different methods.

JHEP03(2004)049

N_f	Unbroken Global Symmetries	
	SUSY Mass	SUSY-Breaking Mass
$N_f < N_c$	$\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$	$\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$
$N_f = N_c$	$\mathrm{SU}(N_f)_V$	$\mathrm{SU}(N_f)_V$
$N_f = N_c + 1$	$\mathrm{SU}(N_f)_V$	$\mathrm{SU}(N_f - 1) \times \mathrm{SU}(N_f)$ or $\mathrm{SU}(N_f - 1)_V$
$N_c + 2 \leq N_f < \frac{3}{2}N_c$	$\mathrm{SU}(N_f)_V$	$\mathrm{SU}(\widetilde{N}_c)_{R+c} \times \mathrm{SU}(N_c)_R \times \mathrm{SU}(N_f)_L$ or $(L \leftrightarrow R)$
$\frac{3}{2}N_c < N_f < 2N_c$	$\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$	$\mathrm{U}(1)_B$ broken
$2N_c < N_f < 3N_c$	$\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$	$\mathrm{U}(1)_B$ broken?

Table 2: Unbroken subgroup of the original $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B$ global symmetry for SQCD with the addition of either supersymmetric mass terms or soft SUSY-breaking masses. See the text for assumptions made in each case.

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A. Harmonic oscillator example

The two-dimensional isotropic harmonic oscillator is an explicit example which can be used to verify the formalism for treating the chemical potential. We begin with the hamiltonian for the 2D harmonic oscillator and add a chemical potential for the "number" operator $-L_z/\hbar$:

$$H - \mu N = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) - \mu\frac{1}{\hbar}(yp_x - xp_y) \tag{A.1}$$

which corresponds to the lagrangian

$$L = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 - (\omega^2 - \hat{\mu}^2)(x^2 + y^2) - 2\hat{\mu}(x\dot{y} - y\dot{x})\right] \tag{A.2}$$

where $\hat{\mu} \equiv \mu/\hbar$. Writing $z = \frac{1}{\sqrt{2}}(x + iy)$ puts this in the form

$$L = m\left[\dot{z}^*\dot{z} - (\omega^2 - \hat{\mu}^2)z^*z + i\hat{\mu}(z^*\dot{z} - z\dot{z}^*)\right] \tag{A.3}$$

which (apart from the factor of m) is the lagrangian for a $(0 + 1)$-dimensional charged scalar field with "mass" ω and nonzero chemical potential.

Apart from the similarity between this system and the field theory that we are interested in, the virtue of this example is that we can compare three separate computations for the partition function. First, one can perform the sum over states in the grand canonical ensemble directly. This is most easily done by writing the hamiltonian in terms of two sets of creation and annihilation operators, $H = (\hbar\omega - \mu)c^\dagger c + \frac{1}{2} + (\hbar\omega + \mu)d^\dagger d + \frac{1}{2}$, where $c = \frac{1}{\sqrt{2}}(a_x + ia_y)$ and $d = \frac{1}{\sqrt{2}}(a_x - ia_y)$. The partition function then becomes simply the product of two independent harmonic oscillators with frequencies offset by $\pm\mu/\hbar$:

$$\mathcal{Z} = \sum e^{-\beta(H-\mu N)} = \frac{1}{2\sinh\frac{\beta(\hbar\omega-\mu)}{2}} \frac{1}{2\sinh\frac{\beta(\hbar\omega+\mu)}{2}} . \tag{A.4}$$

The same result is obtained by computing the grand canonical partition function by an "imaginary time" path integral:

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}x\mathcal{D}y\ e^{-\frac{m}{2\hbar}\int_0^{\beta\hbar} d\tau\left[\dot{x}^2+\dot{y}^2+(\omega^2-\hat{\mu}^2)(x^2+y^2)+2i\hat{\mu}(x\dot{y}-y\dot{x})\right]} \tag{A.5}$$

where here $\dot{x} = \partial_\tau x$.

We can also perform the real-time path integral computation of the quantum mechanical propagator, namely

$$\langle z_f,\, t_f | z_i,\, t_i\rangle = \int \mathcal{D}z^*\mathcal{D}z\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} m\left[\dot{z}^*\dot{z}-(\omega^2-\hat{\mu}^2)z^*z+i\hat{\mu}(z\dot{z}^*-z^*\dot{z})\right]} \tag{A.6}$$

$$= e^{iS_c/\hbar}\frac{m\omega}{\pi i\hbar\sin(\omega(t_f-t_i))} \tag{A.7}$$

where

$$S_c = \frac{m\omega}{\sin\omega T}\left[i(z_f^* z_i - z_f z_i^*)\sin\mu T - (z_f^* z_i + z_f z_i^*)\cos\mu T + (|z_f|^2 + |z_i|^2)\cos\omega T\right] \tag{A.8}$$

is the action over the classical path, and $T = t_f - t_i$. Taking $T = -i\hbar\beta$ and $z_f = z_i = z$ followed by integration over z gives the grand canonical partition function:

$$\mathcal{Z} = \int dz\langle z|e^{-\beta(H-\mu N)}|z\rangle = \frac{1}{2\sinh\frac{\beta(\hbar\omega-\mu)}{2}} \frac{1}{2\sinh\frac{\beta(\hbar\omega+\mu)}{2}} . \tag{A.9}$$

This matches the direct computation, verifying the form of the effective lagrangian in Minkowski space. This exercise demonstrates how the μ^2 term in the lagrangian is necessary to produce the correct result from a path integral calculation.

B. Exact results for soft masses

The following is a review of [10] and [12]. We begin again with the SQCD lagrangian renormalized at a UV scale μ_{UV},

$$\mathcal{L}_{UV} = \int d^4\theta \sum_r \mathcal{Z}_r(\mu_{UV})Q_r^\dagger Q_r + \int d^2\theta S(\mu_{UV})\left(\mathcal{W}_\alpha\mathcal{W}^\alpha + \mathrm{h.c.}\right), \tag{B.1}$$

where the index r labels the quark representations. SUSY breaking effects can be incorporated by promoting the couplings $\mathcal{Z}$ and S to real and chiral superfields respectively as follows[8]

$$\begin{aligned}\mathcal{Z}_r &= Z_r\left[1-\theta^2\bar\theta^2\widetilde{m}^2\right]\\ S &= \frac{1}{g^2}\left(1+\theta^2\frac{m_\lambda}{2}\right).\end{aligned} \tag{B.2}$$

eq. (B.2) incorporates a universal soft mass $\widetilde{m}$ as well as a possible gluino mass m_λ.

The theory possesses a spurious $\mathrm{U}(1)_A$ symmetry under which

$$Q_r \to Q_r e^{A} \qquad \mathcal{Z}_r \to \mathcal{Z}_r e^{-(A+A^\dagger)} \tag{B.3}$$

and the gauge coupling transforms anomalously

$$S \to S - \frac{N_f}{8\pi^2}A\,. \tag{B.4}$$

The main point of [10] and [12] is that this symmetry is respected by the strong dynamics that lead to confinement. Since $\mathcal{Z}$ transforms as a gauge field under this $\mathrm{U}(1)_A$, it's couplings to the low energy degrees of freedom are determined by this symmetry. This in turn determines the IR soft mass, which is the highest component of the superfield $\mathcal{Z}$. Furthermore, any physical quantity, related to either the UV or the IR degrees of freedom, should be invariant under this symmetry and also RG invariant. It is therefore worthwhile to construct a $\mathrm{U}(1)_A$ and RG invariant object from S and $\mathcal{Z}$ since it will have physical meaning in both descriptions. The only such object is

$$I = \Lambda_h^\dagger \mathcal{Z}^{\frac{2N_f}{b_0}}\Lambda_h \tag{B.5}$$

where $\Lambda_h = \mu_{UV}e^{-8\pi^2 S(\mu_{UV})/b_0}$ is the holomorphic dynamical scale and $b_0 = 3N_c - N_f$ is the one loop beta-function coefficient.

Indeed, in Reference [10] it was shown that for UV-free theories the various components of I are related simply to the physical dynamical scale and the bare soft terms

$$\begin{aligned}[I]_{\theta=\bar\theta=0} &= \Lambda^2\\ [\ln I]_{\theta^2} &= \lim_{\mu_{UV}\to\infty}\frac{16\pi^2}{b_0}\frac{m_\lambda}{g^2}\\ [\ln I]_{\theta^2\bar\theta^2} &= -\frac{2N_f}{b_0}\lim_{\mu_{UV}\to\infty}\widetilde{m}_r^2\,.\end{aligned} \tag{B.6}$$

To find the effects of soft masses in the low energy theory, all we need to do is write down the most general Kähler potential allowed by the symmetries, including the anomalous $\mathrm{U}(1)_A$, and determine the coupling of $\mathcal{Z}$ to the kinetic terms. Since we only need squark soft masses to stabilize the theory in the UV we will not add a gaugino mass.

[8]If there is no superpotential we can choose to absorb the θ^2 and $\bar\theta^2$ terms of $\mathcal{Z}$ into the gluino mass through the anomalous symmetry of Equations (B.3) and (B.4).

JHEP03(2004)049

For example, consider the case with $N_f = N_c + 1$. The Kähler potential can be expanded around the origin and is constrained by the symmetries and RG invariance to be

$$K = c_M \frac{\mathcal{Z}^2}{I} \operatorname{Tr} M^\dagger M + c_B \frac{\mathcal{Z}^{N_c}}{I^{N_c-1}} B_i^\dagger e^{N_c V_B} B_i + c_{\overline{B}} \frac{\mathcal{Z}^{N_c}}{I^{N_c-1}} \overline{B}_i^\dagger e^{-N_c V_B} \overline{B}_i + \cdots \tag{B.7}$$

where V_B is the background $\mathrm{U}(1)_B$ gauge field. Note that since this is an expansion around the origin, the potential we will derive will only be valid for VEVs close to the origin that are much smaller than Λ.

The soft masses for the mesons and baryons at some IR scale μ_{IR} will in general depend on the SUSY breaking VEVs of the spurion fields $\mathcal{Z}$, I and V_B as well as $\mathcal{O}(1)$ numerical coefficients c_M, c_B and $c_{\overline{B}}$. However, in [10] it was shown that due to the IR-freedom of the theory the soft masses don't depend on the coefficients c_M, c_B, or $c_{\overline{B}}$ in the deep IR, $\mu_{IR} \to 0$. Using the components of $\mathcal{Z}$ in eq. (B.2), I in eq. (B.6) and V_B in eq. (3.2) and performing the $d^4\theta$ integral gives the soft masses for the canonically normalized mesons and baryons. Altogether we have:

$$\widetilde{m}_M^2 = \frac{2N_c - 4}{2N_c - 1}\widetilde{m}^2 \qquad \text{and} \qquad \widetilde{m}_{B,\overline{B}}^2 = \frac{2 - N_c}{2N_c - 1}\widetilde{m}^2 - N_c^2\mu^2\,. \tag{B.8}$$

Notice that for $N_c \geq 3$ the meson mass-squared is positive, but the baryons are tachyonic.

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